Solutions of the Equation of Motion with Absorption for some common sources

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Summary

Dispersion resulting from absorption in the propagation medium has been included in the approximations of solutions of the equation of motion for some common sources (a directed point force, double-couple-without-moment forces and a shear-dislocation force) by replacing the velocity or slowness with the complex version. A velocity-frequency relation in the form of $\frac{1}{v(\omega)} \approx \frac{1}{v(\omega_{0r})}\left[1-\frac{1}{\pi Q}\,ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q}\right]$ has been used. These approximations match very well with the exact numerical results, and the anelastic waveforms have a different shape than the elastic ones.

Introduction

Over the past few decades, many kinds of the solutions of typical equation of motion in a perfectly elastic medium have been derived and also have been widely recognized. However, in fact, we can hardly find an ideal medium in practical applications. In real materials, wave energy is absorbed due to internal friction or anelasticity. Absorption is frequency dependent, i.e., different frequencies are absorbed by different amounts. One consequence of this is that the waveform changes with distance travelled. Therefore, developing new solutions of the EOM (equation of motion) with absorption is a very meaningful thing.

Normally, it's assumed that absorption is a linear phenomenon, so that Fourier analysis can be used. Also, we are accustomed to using the quality factor Q to express the effect of absorption on the waveform. Q in the earth's mantle is nearly independent of frequency, and for a perfectly elastic medium, Q=∞.

Theory and Method

We can start with some well-known solutions of the EOM to extend them to the case of absorption. For example, the solution of the EOM for a directed point force (see, e.g., Aki and Richards I, 2002, p. 68, and Achenbach, 1973, p. 96) is

$$u_i(\mathbf{x},t)=\frac{(3\gamma_i\gamma_j\text{-}\delta_{ij})}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau s(t\text{-}\tau)\,d\tau+\frac{\gamma_i\gamma_j}{4\pi\rho\alpha^2 r}s\left(t\text{-}\frac{r}{\alpha}\right)+\frac{(\delta_{ij}\text{-}\gamma_i\gamma_j)}{4\pi\rho\beta^2 r}s(t\text{-}\frac{r}{\beta})$$

$$\equiv\ u_i{}^N+u_i{}^P+u_i{}^S,\qquad i=1,2,3 \tag{1}$$

where u_i is the i-th component of displacement, x is the vector from the origin to the observation point (i.e., the location of the seismometer), and $\boldsymbol{\gamma}\equiv\mathbf{x}/r$ is a unit vector in the direction of x (i.e., $\gamma_n=x_n/r$ is the direction cosine between x and the x_n axis). Since the wave travels from the origin to x, the vector x and $\boldsymbol{\gamma}$ are in the direction of wave propagation. j indicates the direction of the point force. The first term, $u_i{}^N$, is the near-field term. It dominates over the other two terms at small values of distance r. The second term, $u_i{}^P$, is the far-field P wave term. The third term, $u_i{}^S$, is the far-field S wave term.

Since absorption is frequency dependent, if we want to add its influence into the solution, firstly, we should do the Fourier transform of the solution (1) to convert it into frequency domain. Then, modify this Fourier-transformed solution to include absorption. Absorption can be included in wave motion by making the slowness complex (with the angular frequency being real), then

$$\frac{1}{v}=\frac{1}{v_0}\left(1+\frac{i}{2Q_v}\right) \tag{2}$$

, where v_0 is the wave speed in elastic medium.

However, anelasticity of the earth causes physical dispersion of seismic waves. The significant effect of physical dispersion on surface wave phase and group velocities and free oscillation periods has been dicussed many times, e.g. Jeffreys [1965], Davies [1967], Randall [1976], Kanamori and Anderson [1977], so a velocity-frequency relation in the form of

$$\frac{1}{v(\omega)} \approx \frac{1}{v(\omega_{0r})}\left[1-\frac{1}{\pi Q} ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q}\right] \tag{3}$$

should be used for correcting the effect of dispersion arising from anelasticity, where v is the phase velocity, ω is the angular frequency, ω_{0r} is the reference angular frequency.

If we assume the point force is

$$s(t)=\frac{2atA}{\pi(a^2+t^2)^2} \text{ (in time domain)} \quad \text{and} \quad \bar{s}(\omega)=Ai\omega e^{-a|\omega|} \text{(in frequency domain)} \tag{4}$$

, where a is a positive constant has the units of seconds and A is also a constant has the units of mass *length.

Subsititute (3) and (4) into the Fourier-transformed solution (1), and then we will get a solution of the EOM for a directed point force with dispersion in the frequency domain, which should be transformed back to the time domain by using inverse Fourier transform.

Examples

Still consider about the directed point force we mentioned above. If the point force is in the x direction and we only talk about the x component of displacement, which means i=1 and j=1, after all the calculations, we can get an approximation of the solution of the EOM for absorption with dispersion in the time domain (5).

$$\begin{aligned}
u_1(t) = & \frac{A}{2\pi^2\rho r^3}\left[-\left(\cos(F_\beta)+F_\beta \sin(F_\beta)\right)\tan^{-1}\left(\frac{G_\beta}{K_\beta}\right)+\left(\cos(F_\alpha)+F_\alpha\sin(F_\alpha)\right)\tan^{-1}\left(\frac{G_\alpha}{K_\alpha}\right)\right] \\
& + \frac{A}{2\pi^2\rho r^3}\frac{\left(\frac{r}{\beta(\omega_{0r})}-\frac{F_\beta}{\omega_{0r}}\right)\left(K_\beta\cos(F_\beta)-G_\beta\sin(F_\beta)\right)-\left(K_\beta-a\right)\left(K_\beta\sin(F_\beta)+G_\beta\cos(F_\beta)\right)}{{K_\beta}^2+{G_\beta}^2} \\
& - \frac{A}{2\pi^2\rho r^3}\frac{\left(\frac{r}{\alpha(\omega_{0r})}-\frac{F_\alpha}{\omega_{0r}}\right)\left(K_\alpha\cos(F_\alpha)-G_\alpha\sin(F_\alpha)\right)-\left(K_\alpha-a\right)\left(K_\alpha\sin(F_\alpha)+G_\alpha\cos(F_\alpha)\right)}{{K_\alpha}^2+{G_\alpha}^2} \\
& + \frac{A}{4\pi^2\rho r}\frac{\left[\frac{\sin(F_\alpha)}{Q_\alpha}-\left(1-\frac{2}{\pi Q_\alpha}\right)\cos(F_\alpha)\right]2K_\alpha G_\alpha-\left[\frac{\cos(F_\alpha)}{Q_\alpha}+\left(1-\frac{2}{\pi Q_\alpha}\right)\sin(F_\alpha)\right]\left({K_\alpha}^2-{G_\alpha}^2\right)}{\alpha^2(\omega_{0r})\left({K_\alpha}^2+{G_\alpha}^2\right)^2} \\
& - \frac{A}{2\pi^2\rho r^2}\frac{F_\alpha\left(\cos(F_\alpha)G_\alpha+\sin(F_\alpha)K_\alpha\right)}{\alpha(\omega_{0r})\left({K_\alpha}^2+{G_\alpha}^2\right)}
\end{aligned} \tag{5}$$

where $F_v=\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}$, $K_v=\frac{r}{2Q_v v(\omega_{0r})}+a$, $G_v=\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t$, and $\alpha, \beta, Q_\alpha\ and\ Q_\beta$ are P velocity, S velocity, Q for P waves and Q for S waves.

And we can also obtain an exact formula by using the relation (2) for $u_1(t)$ which includes absorption but not dispersion.

$$u_1(t) = \frac{1}{2\pi^2\rho r^2}\left(\frac{\frac{1}{\beta_0}\left(a+\frac{t}{2Q_\beta}\right)}{\left(a+\frac{r}{2Q_\beta\beta_0}\right)^2+\left(\frac{r}{\beta_0}-t\right)^2}-\frac{\frac{1}{\alpha_0}\left(a+\frac{t}{2Q_\alpha}\right)}{\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)^2+\left(\frac{r}{\alpha_0}-t\right)^2}\right)$$
$$+\frac{1}{2\pi^2\rho r^3}\left(\tan^{-1}\frac{\frac{r}{\alpha_0}-t}{a+\frac{r}{2Q_\alpha\alpha_0}}-\tan^{-1}\frac{\frac{r}{\beta_0}-t}{a+\frac{r}{2Q_\beta\beta_0}}\right)-\frac{1}{2\pi^2\rho{\alpha_0}^2}\frac{\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\left(\frac{r}{\alpha_0}-t\right)}{\left(\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)^2+\left(\frac{r}{\alpha_0}-t\right)^2\right)^2}$$
$$-\frac{1}{4\pi^2\rho{\alpha_0}^2Q_\alpha}\frac{\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)^2-\left(\frac{r}{\alpha_0}-t\right)^2}{\left(\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)^2+\left(\frac{r}{\alpha_0}-t\right)^2\right)^2} \quad (6)$$

To verify the accuracy of the approximation (5), we created a program by Matlab to compute the exact results with velocity dispersion numerically. Compare the approximation and exact numerical result in Fig.1, and we can see they coincide well. At small values of distance, e.g., r=1km, the near-field term and far-field term can be distinguished easily. While, for a relative long distance, e.g., r=20km, the far-field term dominates over the near-field term.

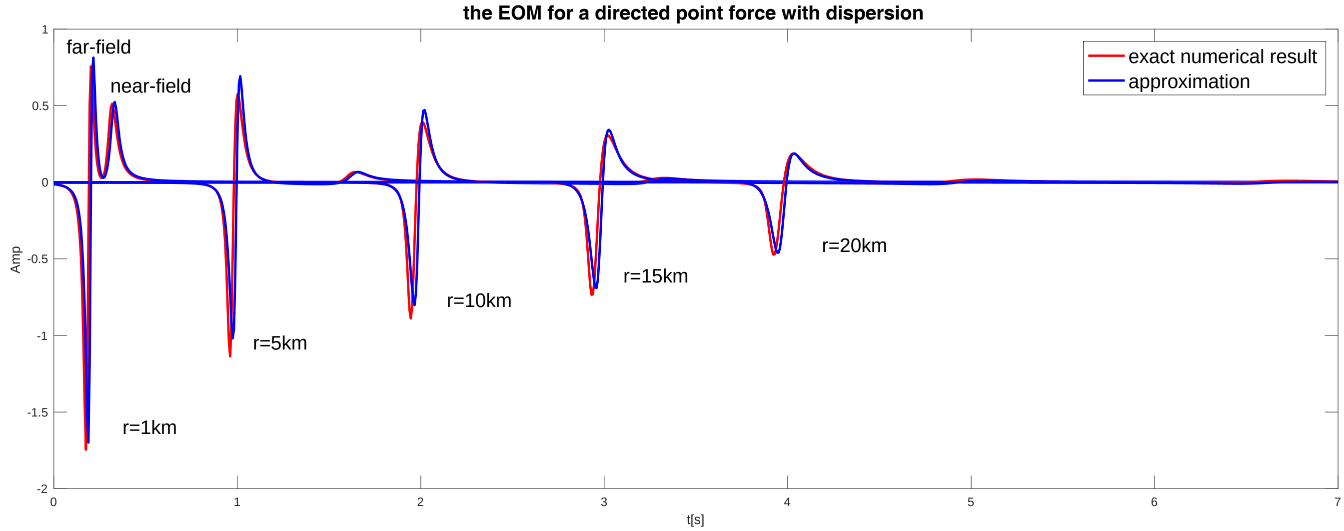

Fig.1 When distance between source and receiver is 1km, 5km, 10km, 15km and 20km, the approximation and real numerical result match with each other very well. The amplitude of the waveforms for different distance r from 1km to 20km has been multiplied by 1, 7, 20, 40 and 50, respectively. The parameters used in this figure are $\alpha = 5km/s$, $\beta = 3km/s$, $Q_\alpha = 30$, $Q_\beta = 15$, $a = 0.02$, $\omega_{0r} = 20$, $A = 1$, $\rho = 1$.

Since we have got the anelastic waveforms, the next step is to find the differences between the anelastic waveforms and the elastic ones. The result is showed in Fig.2. The red curves are the exact numerical results of anelastic waveforms, and the black curves are elastic waveforms. The effect of dispersion becomes more and more obvious as the propagation distance increases.

The double-couple-without-moment is now accepted as the best body-force model of an earthquake source. It's composed of two single couples with opposite moments. The solution of the EOM of it can be found in Aki and Richards, 2002, eq. 4.29, p. 77. If we consider a vertical fault coinciding with the xz plane, and the only non-zero components of the moment tensor are then $M_{12} = M_{21} = M_0$ (M is a moment tensor, and we still assume $M_0(t) = \frac{2atA}{\pi(a^2+t^2)^2}$ and $\overline{M_0}(\omega) = Ai\omega e^{-a|\omega|}$), applying (4.29) in Aki and Richards to obtain the x-component displacement due to the double-couple with velocity dispersion gives:

$$u_1(t) = \frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4}\left(H_\alpha - H_\beta\right) + \frac{A(2+3\cos 2\theta)\sin\theta}{2\pi^2\rho r^2}I_\alpha - \frac{A(3+6\cos 2\theta)\sin\theta}{4\pi^2\rho r^2}I_\beta$$

$$+\frac{A(1+\cos 2\theta)\sin\theta}{4\pi^2\rho r}J_\alpha - \frac{A\cos 2\theta\sin\theta}{4\pi^2\rho r}J_\beta \tag{7}$$

where, θ is the angle from x axis to y axis, and

$$H_v = (\cos(F_v) + F_v\sin(F_v))\tan^{-1}\left(\frac{G_v}{K_v}\right) - \frac{\left(\frac{r}{v(\omega_{0r})} - \frac{F_v}{\omega_{0r}}\right)(K_v\cos(F_v) - G_v\sin(F_v)) - (K_v - a)(K_v\sin(F_v) + G_v\cos(F_v))}{{K_v}^2 + {G_v}^2}$$

$$I_v = \frac{\left[\frac{\sin(F_v)}{Q_v} - \left(1-\frac{2}{\pi Q_v}\right)\cos(F_v)\right]2K_vG_v - \left[\frac{\cos(F_v)}{Q_v} + \left(1-\frac{2}{\pi Q_v}\right)\sin(F_v)\right]\left({K_v}^2 - {G_v}^2\right)}{v^2(\omega_{0r})\left({K_v}^2+{G_v}^2\right)^2} - \frac{2F_v(\cos(F_v)\,G_v + \sin(F_v)\,K_v)}{rv(\omega_{0r})\left({K_v}^2+{G_v}^2\right)}$$

$$J_v = \frac{\left[\left(1-\frac{3}{\pi Q_v}\right)\cos(F_v) - \frac{3\sin(F_v)}{2Q_v}\right]\left(2{K_v}^3 - 6K_v{G_v}^2\right) - \left[\left(1-\frac{3}{\pi Q_v}\right)\sin(F_v) + \frac{3\cos(F_v)}{2Q_v}\right]\left(6{K_v}^2G_v - 2{G_v}^3\right)}{v^3(\omega_{0r})\left({K_v}^2+{G_v}^2\right)^3}$$

$$+\frac{3F_v\left[\cos(F_v)\left({K_v}^2-{G_v}^2\right) - \sin(F_v)\,2K_vG_v\right]}{rv^2(\omega_{0r})\left({K_v}^2+{G_v}^2\right)^2}$$

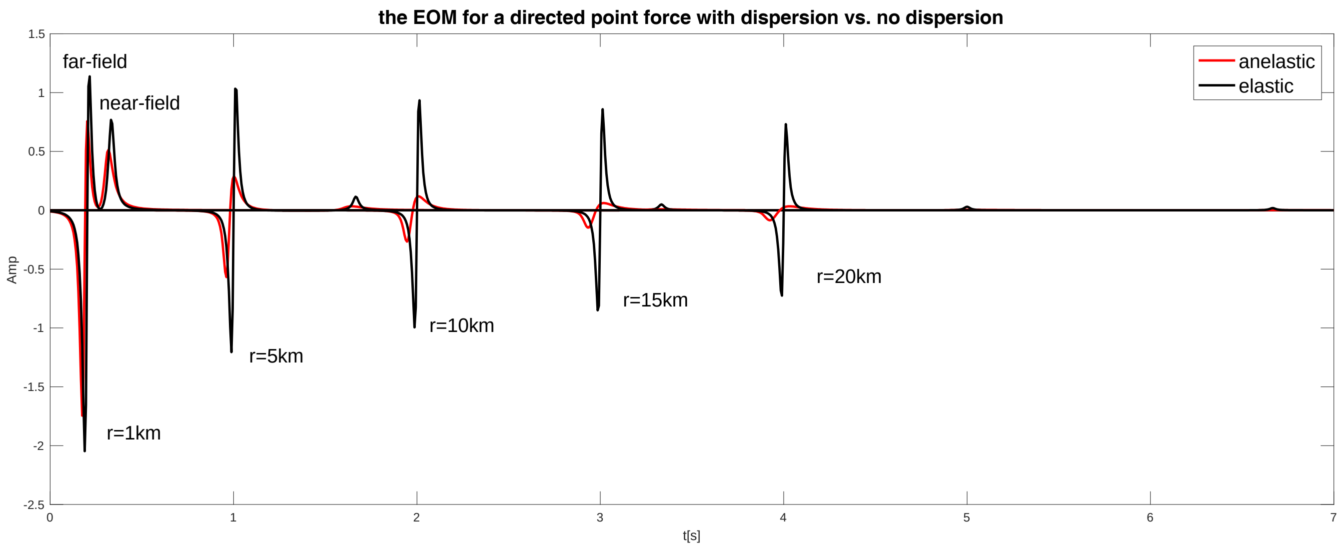

Fig.2 The amplitude of the waveforms for different distance r from 1km to 20km has been multiplied by 1, 3.5, 6, 8 and 9, respectively. In addition to using elastic waveforms instead of the approximation of anelastic waveforms, the other conditions in Fig.2 are the same as in Fig.1.

The solution of the EOM with absorption for a shear dislocation force we have got is very similar to the equation (7), and both of the solutions with absorption for double-couple-without-moment forces and a shear dislocation force work well (match well with exact numerical results and can tell the difference between anelastic waveforms and elastic ones).

Conclusions

To correct the effect of physical dispersion arising from anelasticity, $\frac{1}{v(\omega)} \approx \frac{1}{v(\omega_{0r})}\left[1 - \frac{1}{\pi Q}\ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q}\right]$ has been introduced into the solutions of the EOM for some common sources. In seismic frequency band, Q is nearly independent of frequency. And normally, it is assumed that absorption is a linear phenomenon, so that Fourier analysis can be used. We have shown in our results that when Q is relatively small, the effect of absorption on waveforms cannot be ignored or assumed to be negligible. It’s necessary to include it into solutions of the Equation of Motion.

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