Seismic data reconstruction via fast and memory efficient Singular Spectrum Analysis

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Summary

We study the computational complexity of Singular Spectrum Analysis and present a fast and memory efficient implementation that does not require constructing Hankel trajectory matrices. The key is to replace the singular value decomposition of the Hankel matrix by the randomized QR decomposition. We also present a new strategy in which anti-diagonal averaging of the Hankel matrix is efficiently computed via convolution. The new algorithm significantly decreases the computational cost and memory requirement of Singular Spectrum Analysis data recovery. We test the effectiveness of the method through synthetic and real data examples.

Introduction

During the last decade, a family of rank-reduction techniques based on Cadzow filtering, also called Singular Spectrum Analysis, has been developed for the enhancement of the signal-to-noise ratio and reconstruction of seismic records (Sacchi, 2009; Trickett and Burroughs, 2009; Trickett et al., 2010). For a 2D seismic gather $D(\omega,x)$ in the frequency-space domain, at a given monochromatic frequency ω_0, the frequency slice can be denoted as $D(\omega_0,x) = [D_1,D_2,\cdots,D_N]^T$, where N is the total number of traces. Applying the SSA algorithm to seismic data entails the following three steps:

- We first embed $D(\omega_0,x)$ into a Hankel structured trajectory matrix $\mathbf{H}$ as follows

$$\mathbf{H} = \begin{bmatrix} D_1 & D_2 & \cdots & D_M \\ D_2 & D_3 & \cdots & D_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ D_L & D_{L+1} & \cdots & D_N \end{bmatrix}, \tag{1}$$

  where $L+M-1=N$ and $L \le M$. The desired rank of matrix $\mathbf{H}$ is K if the data consists K dipping events (Oropeza and Sacchi, 2011). Incoherent noise and missing observations will increase the rank of the Hankel matrix.

- Therefore, the second step of SSA entails finding a low-rank approximation of the Hankel trajectory matrix. This is usually done by the Singular Value Decomposition (Golub and van Loan, 1996)

$$[\mathbf{U},\mathbf{S},\mathbf{V}] = \mathrm{SVD}[\mathbf{H}], \tag{2}$$

  where $\mathbf{U}$ and $\mathbf{V}$ are orthonormal matrices and $\mathbf{S}$ is a diagonal matrix. Note that this step can be extremely expensive for large matrices. The computational complexity of the singular value decomposition is approximately $\mathcal{O}(L^2M+LM+M^3)$ (Golub and van Loan, 1996). A new set of singular values $\hat{\mathbf{S}}$ are computed via

$$\begin{aligned} \hat{\mathbf{S}}_{l,l} &= \mathbf{S}_{l,l} \quad & l \le K \\ \hat{\mathbf{S}}_{l,l} &= 0 \quad & l > K \end{aligned} \tag{3}$$

  The low-rank approximation of the Hankel matrix is then computed via

$$\hat{\mathbf{H}} = \mathbf{U}\hat{\mathbf{S}}\mathbf{V}. \tag{4}$$

The regrouping of the Hankel matrix yields a complexity $\mathscr{O}(L^2K)$.

- In the last step, anti-diagonal averaging of the reduced-rank matrix is applied to recover the filtered signal. In other words, filtered data can be recovered via

$$\hat{D}_i = \begin{cases} \frac{1}{i}\sum_{j=1}^{i}\hat{H}_{j,i-j+1} & 1 \leq i \leq M\,, \\ \frac{1}{M}\sum_{j=1}^{M}\hat{H}_{j,i-j+1}, & M \leq i \leq L\,, \\ \frac{1}{N-i+1}\sum_{j=i-L+1}^{M}\hat{H}_{j,i-j+1}, & L \leq i \leq N\,, \end{cases} \tag{5}$$

  where i denotes the trace indices. Anti-diagonal averaging requires $\mathscr{O}(N)$ multiplications and $\mathscr{O}(ML)$ sums (Korobeynikov, 2010).

Compared with other filtering methods, SSA increases the redundancy of data by forming Hankel matrices. The latter enhances the capability for noise suppression while preserving the desired signal. However, the costs of the SVD and the storage of the trajectory matrices make the algorithm unfeasible for large multidimensional seismic data (Gao et al., 2013). We present a fast and memory efficient implementation for SSA that does not require saving Hankel matrices. We use randomized QR decomposition for fast matrix rank reduction (Chiron et al., 2014; Cheng and Sacchi, 2016). Then the final anti-diagonal averaging of the Hankel matrix is computed efficiently via a convolution algorithm (Korobeynikov, 2010). We also extend the method to mutli-dimensional cases where the low-rank approximation of block Hankel matrices is adopted (Gao et al., 2013). A real data example is utilized to test the performance of the proposed algorithm.

Theory

Firstly, We propose to use the randomized QR decomposition as an alternative to the SVD. Instead of applying SVD to the Hankel matrix, a random projection is first performed

$$\mathbf{M} = \mathbf{H}\Omega, \tag{6}$$

where Ω denotes a random set that is composed by k independent vectors ($k \ll M$). The random projection shrinks the size of Hankel matrix ($L \times M$) to a much smaller matrix ($L \times k$) while keeping as much variability as possible. Then an economic-size QR decomposition is applied to the matrix $\mathbf{M}$

$$[\mathbf{Q},\mathbf{R}] = \mathrm{qr}(\mathbf{M}). \tag{7}$$

We point out that the QR decomposition is a very stable algorithm that permits to compute orthonormal basis $\mathbf{Q}$ ($L \times k$). The low-rank approximation can be computed by projecting $\mathbf{H}$ onto the orthonormal basis $\mathbf{Q}$

$$\hat{\mathbf{H}} = \mathbf{Q}(\mathbf{Q}^{*}\mathbf{H})\,. \tag{8}$$

Since the QR decomposition is applied on a shrink-sized matrix, the randomized QR decomposition shows promising improvements in computing the low-rank approximation of a given matrix (Cheng and Sacchi, 2016). In addition, randomized QR decomposition is less stringent on the choice of rank (number of dips)(Chiron et al., 2014). The latter is very important as we usually do not have prior information about the rank of the seismic data.

Secondly, we show the random projection in randomized QR decomposition can be computed using a fast Hankel matrix-vector product. The idea is to embed the Hankel matrix into a circulant matrix and then use Fast Fourier transform to compute matrix vector multiplications (O'Leary and Simmons, 1981). A circulant matrix $\mathbf{C}$ times a vector $\mathbf{x}$ is computed via the Fast Fourier Transform

$$\mathbf{Cx} = \mathscr{F}^{-1}(\mathscr{F}(\mathbf{c}) \circ \mathscr{F}(\mathbf{x}))\,, \tag{9}$$

where $\mathbf{c}$ is the first column of the circulant matrix and $\circ$ denotes the Hadamard (element-wise) product. We refer the readers to Gao et al. (2013) where the authors discussed in details the embedding of a Hankel matrix into a circulant matrix. A Hankel matrix can be easily converted to a Toeplitz matrix by reversing the columns. We conveniently adopt the algorithm in Gao et al. (2013) given the following relation

$$\mathbf{y} = \mathbf{H}\mathbf{x} = \mathbf{T}\mathbf{z}, \tag{10}$$

where **T** denotes a Toeplitz matrix and $\mathbf{x}$ is a vector (Korobeynikov, 2010). The vector $\mathbf{z}$ is obtained by reversing the order of the entries of $\mathbf{x}$. The vector $\mathbf{y}$ is the result of the multiplication. Equation 6 can be treated as k Hankel matrix-times-vector products and thus it can be computed efficiently as no Hankel matrices are formed.

Finally, We show that the anti-diagonal averaging of Hankel matrix can be efficiently computed via convolution. To clearly demonstrate the method, we assume that the desired rank of the Hankel trajectory matrix equals 1. Equation 8 reduces to

$$\hat{\mathbf{H}} = \mathbf{q}_1 \mathbf{t}_1 \,, \tag{11}$$

where $\mathbf{t}_1 = \mathbf{q}_1^* \mathbf{H}$ and $\mathbf{q}_1$ denotes the first row of the matrix $\mathbf{Q}$ from the QR decomposition. Apparently $\mathbf{t}_1$ can be computed via the fast Hankel matrix-vector product. Combining Equation 5 and Equation 11 yields the following expression

$$\hat{D}_i = \begin{cases} \frac{1}{i}\sum_{j=1}^{i} q_{1_j} t_{1_{i-j+1}}, & 1 \le i \le M \\ \frac{1}{M}\sum_{j=1}^{M} q_{1_j} t_{1_{i-j+1}}, & M \le i \le L \\ \frac{1}{N-i+1}\sum_{j=i-L+1}^{M} q_{1_j} t_{1_{i-j+1}}, & L \le i \le N \end{cases}, \tag{12}$$

or equivalently,

$$\hat{D}_i = w_i \sum_{j=1}^{N} q_{1_j} t_{1_{i-j+1}}, \tag{13}$$

where w_i denotes constants that are determined by the size of Hankel matrix and are computed in advance. We can rewrite Equation 13 in its vector form

$$\hat{\mathbf{D}} = \mathbf{w} \circ [s_1(\mathbf{q}_1 * \mathbf{t}_1)], \tag{14}$$

where $\mathbf{q}_1 * \mathbf{t}_1$ denotes the convolution that can be efficiently computed using the Fast Fourier Transform. We can repeat the process to compute each eigenimage corresponding to each desired singular value of the rank-K approximation. The filtered data equals the summation of the K eigenimages

$$\hat{\mathbf{D}} = \mathbf{w} \circ [(\mathbf{q}_1 * \mathbf{t}_1) + (\mathbf{q}_2 * \mathbf{t}_2) + \cdots + (\mathbf{q}_K * \mathbf{t}_K)]. \tag{15}$$

The computational complexity reduces to $\mathscr{O}(N\log(k))$. The full algorithm is summarized in algorithm (1), where fast_multiply denotes the fast Hankel matrix vector product.

Example

The following example involves the reconstruction of a real prestack 5D volume via the proposed fast and memory efficient SSA method. We extracted a small patch of an orthogonal land survey from the Western Canadian Sedimentary Basin. After binning, the seismic traces were assigned to the midpoint-offset grid which has dimensions $16 \times 18 \times 12 \times 12$. Around 40% of the 4D grid do not contain observations. We selected a time window in the interval $900 - 1250$ msec that corresponds to 351 samples. Figure 1 (a) shows a subset of the data prior to reconstruction. This subset is acquired by fixing CMP$_y$ and Offset$_y$. Figure 1 (b) illustrates the reconstruction via the proposed fast SSA method with $\alpha = 0.4$ and $k = 18$.

Algorithm 1 Fast and memory efficient SSA

for $\omega = \omega_{min} : \omega_{max}$ **do**
    $\mathbf{d} = \mathbf{D}(\omega, :)$; $\Omega = \text{rand}(M, k)$ (generate random vectors)
    for $i = 1 : k$ **do**
        $\mathbf{M}(:, i) = \text{fast\_multiply}(\mathbf{d}, \Omega(:, i))$
    end for
    $[\mathbf{Q}, \mathbf{R}] = \text{qr}[\mathbf{M}]$
    for $i = 1 : k$ **do**
        $\mathbf{q} = \mathbf{Q}(:, i)$; $\mathbf{z} = \text{fast\_multiply}(\mathbf{d}, \mathbf{q})$
        $\hat{\mathbf{d}} = \hat{\mathbf{d}} + \text{ifft}(\text{fft}(\mathbf{q}) \circ \text{fft}(\mathbf{z}))$ (convolution)
    end for
    $\hat{\mathbf{D}}(\omega, :) = \hat{\mathbf{d}}$
end for

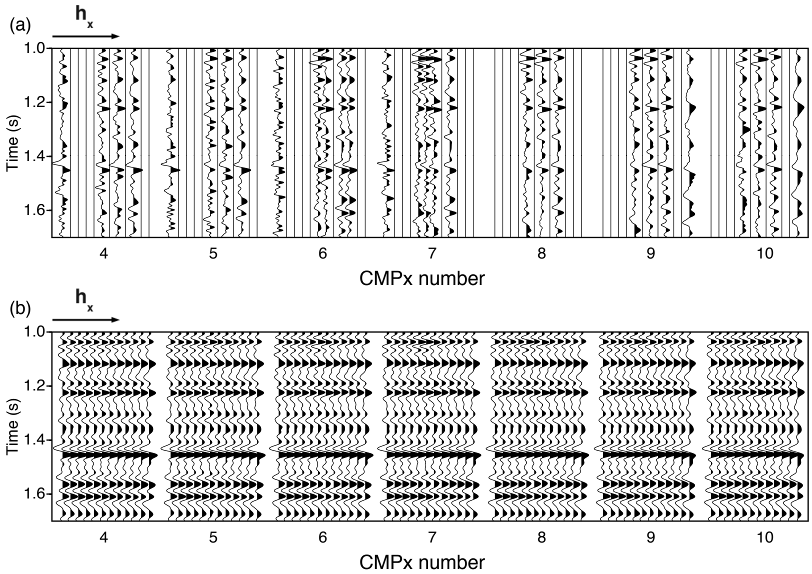

Figure 1: Results of SSA de-noising and reconstruction for a 3D field data. (a) A small patch of field data before reconstruction. (b) De-noised and reconstructed data via the fast SSA algorithm.

Conclusion

This article illustrates a fast and memory efficient implementation of the Singular Spectrum Analysis data recovery algorithm. The method is based on the randomized QR decomposition to perform matrix rank reduction of a shrink-sized matrix. Important savings for large-scale problems are attainable via this technique. To avoid the construction of Hankel matrices in the SSA algorithm, we proposed to adopt fast Hankel matrix-vector product and a fast convolution for the final anti-diagonal averaging. Both Hankel matrix-vector product and the fast anti-diagonal averaging are computed via Fast Fourier Transform. The proposed method significantly improves the computational efficiency of SSA. The proposed fast and memory efficient SSA can be extended to multi-dimensional scenarios and utilized to de-noise and reconstruct 5D seismic volumes.

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