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Regular grids travel time calculation – Fast marching with an adaptive stencils approach

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Summary

The Finite Difference Eikonal solver provides an efficient algorithm for grid travel time calculation. Since both efficiencies and accuracy are important in exploration seismic application, we present a directional depravity method that can accommodate different directions of wave propagation. The new algorithm is implemented on a Cartesian coordinate system with a simple first order finite difference scheme and therefore, it can not only improve accuracy but can also retain efficiency.

Introduction

Since Vidale (1988) introduced the finite difference Eikonal solver into seismology, this algorithm has undergone much development and has now become a relatively mature method that is widely used in seismic applications such as tomography and migration. While the fast marching technique has made most of the gains in efficiency for the finite difference Eikonal solver (e.g. Sethain and Popovici, 1999), accuracy is still a problem waiting for improvement. Because errors in the finite difference solver could be spread and accumulated, improving accuracy has and continues to be an interesting research topic. Alkhalifah and Formel (2001) implemented a fast matching eikonal solver in a polar coordinate system. The accuracy of this method can only be guaranteed for the case where the source must be at the coordinate's origin and the wavefront is circular from the center of the source position. However when waves propagate through a complicated geological structure, such a circular wavefront may not hold, e.g. refracted wavefront, and therefore, implementation of finite difference in polar coordinates may not have any advantage. High order finite difference schemes have also been used to improve accuracy (e.g. Rickett and Fomel,1999, Ahmed et al, 2011, Gillberg et al. 2012). However, because finite difference itself is a linear approximation and the wavefront is, in general, not a plane wave, especially near the source, a high order scheme in such a case may not be helpful and may further increase computational complexity. A Finite Difference scheme that assumes a local wavefront is generated by a virtual source can handle a curved wavefront (e.g. Zhang et al, 2005). However, this assumption is not valid for a relatively flat wavefront, e.g. when the wavefront is far from the source.
In the following, we first use a simple example to show how the error depends on the direction of wave propagation and then we present a new finite difference scheme that uses a directional derivative for the rectangular grid size and finally we give examples to demonstrate the effect of this algorithm.

A numerical error analysis with a simple example

The 2-D Eikonal equation, governing the traveltimes from a fixed source in isotropic media, has the form

$$\left(\frac{\partial \tau}{\partial x}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2 = \frac{1}{v^2(x,z)} \qquad (1)$$

Here *x*, and *z* are spatial coordinates, τ is the traveltime (eikonal), and *v* is the velocity field. Using finite difference to numerically solving this equation produces first arrival traveltimes.

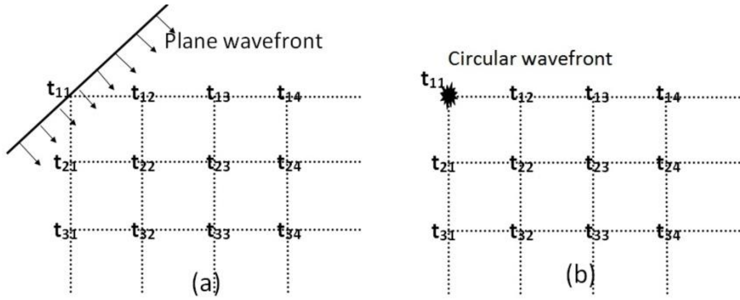


Figure 1. Illustrating (a) a plane wavefront and (b) circular wave from a point source propagating through a grid model.

Numerically solving this equation produces first arrival traveltimes. An example is shown in Figure 1 of a grid model in which a plane wavefront, (Figure 1a) and a circular wavefront, (Figure 1b) are respectively propagating through a regular grid model. Model parameters are set for both velocity and the grid size and grid traveltimes are pre-calculated. With this specific configuration, the traveltime on the grid, say $t_{i,j}$, calculated by three of the most popular finite difference schemes are

Vidale (1988):

$$\left(\frac{t_{i,j}-t_{i-1,j-1}}{\sqrt{2}}\right)^2+\left(\frac{t_{i,j-1}-t_{i-1,j}}{\sqrt{2}}\right)^2=1 \qquad \text{(2a)}$$

Podvin and Lecomte, (1991):

$$\left(\frac{t_{i,j}-t_{i,j-1}}{1}\right)^2+\left(\frac{t_{i,j-1}-t_{i-1,j-1}}{1}\right)^2=1 \qquad \text{(2b)}$$

Rickett, et al (1999, second order scheme):

$$\left(\frac{3t_{i,j}-4t_{i-1,j}+t_{i-2,j}}{2}\right)^2+\left(\frac{3t_{i,j}-4t_{i,j-1}+t_{i,j-2}}{2}\right)^2=1 \qquad \text{(2c)}$$

In order to check the accuracy of these finite difference schemes, we insert known exact travel times into the formulae to see how they satisfy these equations. For the case of the plane wavefront, all of these equations are exact because the traveltime from a plane wave can be linearly interpolated and therefore, the finite difference is accurate. However, when we move to the circular wavefront case, apart from the grids on both horizontal and vertical direction that can be correctly calculated with direct expansion, none of the formulae above can be accurate except equation (2a) which is exact for the grid points that lie along a diagonal direction. Taking $t_{2,2}$ for example, we can see that Vidale's equation can be exactly satisfied while for Popovici's scheme, equation (2b), the left side of the equation is $\left(\sqrt{8}-\sqrt{5}\right)^2+\left(\sqrt{5}-\sqrt{2}\right)^2=1.02$, and the left side of the second order scheme, equation (2c), is 1.1874. Actually, the result of $t_{3,3}$ from equation (2b) is 2.8058, from (2c) is 2.7861 while the true solution is 2.8284 as calculated from equation (2a). Now we examine the case where grid points are off-diagonal. Taking $t_{2,3}$ for example, the left side for (2a) is 0.9656, (2b) is 0.9548 and (2c) is 0.9569. The solutions to $t_{2,3}$ are 3.6103, 3.6341 and 3.5828 for equations (2a), (2b) and (2c) respectively, while the true solution is 3.6056.

The results from the calculations of $t_{2,2}$ and $t_{2,3}$ show some interesting points: firstly, high order finite differences may not necessarily improve accuracy; and secondly, grids on a diagonal direction can be exactly calculated with Vidale's centered scheme. Carefully investigating why Vidale's centered scheme gives an exact solution along the diagonal direction, we find that, if we put the local wavefront coordinate coincident with the diagonal, then equation (2a) is equivalent to ray tracing. Therefore, if we can do a similar thing to all grids then we can expect to calculate traveltime accurately. However, this will effectively come to the same procedure as that of ray tracing. The error resulting from the Finite difference approximation depends on the direction of wave propagation and has been considered in methods for improving accuracy, such as using an auxiliary grid (Cao, et al, 1994) and tetragonal coordinates (Sun and Formel, 1998). However, all of those methods increase the complexity of the calculation which in turn affects efficiency.

Centered FD scheme for a rectangular grid

Since a centered finite difference scheme can be more accurate than a one-sided scheme, we now consider a general rectangular grid with a centered finite difference scheme, which we expect to be more robust in accommodating different directions of wave propagation. If we re-examine Figure 1(b) when considering time at $t_{2,3}$, the propagating direction from source point is 63.4 degree instead of the 45 degree favored by equation 2a. However, if we consider a rectangular grid that consists of points: $t_{1,1}, t_{1,3}, t_{2,1}$ and $t_{2,3}$, then the centered finite difference is carried in the diagonal directions, denoted as $\vec{l}_1$, direction from $t_{1,1}$ to $t_{2,3}$ and $\vec{l}_2$ from $t_{1,2}$ to $t_{2,1}$ respectively. With directional derivative $t_{\vec{l}_1}$ corresponding to direction $\vec{l}_1$ and $t_{\vec{l}_2}$ corresponding to direction $\vec{l}_2$, we have

$$t_{\vec{l}_1} = \overrightarrow{l_1} \cdot \nabla t = t_x \cos\theta + t_y \sin\theta$$
$$t_{\vec{l}_2} = \overrightarrow{l_2} \cdot \nabla t = t_x \sin\theta + t_y \cos\theta$$

where θ is the angle between $\vec{l}_1$ and horizontal direction. Writing this in a matrix form:

$$\begin{pmatrix} t_{\vec{l}_1} \\ t_{\vec{l}_2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} t_x \\ t_y \end{pmatrix} = R\nabla T$$

Thus, we have

$$|\nabla T|^2 = (\nabla T)^T \nabla T = (R^{-1}U)^T * (R^{-1}U) = U^T (RR^T)^{-1} \nabla T$$

$$(RR^T) = \begin{pmatrix} 1 & \cos\emptyset \\ \cos\emptyset & 1 \end{pmatrix}$$

Where $\emptyset$ is angle between $\vec{l}_1$ and $\vec{l}_2$

$$(RR^T)^{-1} = \frac{1}{\sin\emptyset} \begin{pmatrix} 1 & -\cos\emptyset \\ -\cos\emptyset & 1 \end{pmatrix}$$

$$(t_{\vec{l}_1}^2 + t_{\vec{l}_2}^2 - 2 t_{\vec{l}_1} t_{\vec{l}_2} \cos\emptyset) \frac{1}{\sin^2\emptyset} = \frac{1}{v} \qquad (3)$$

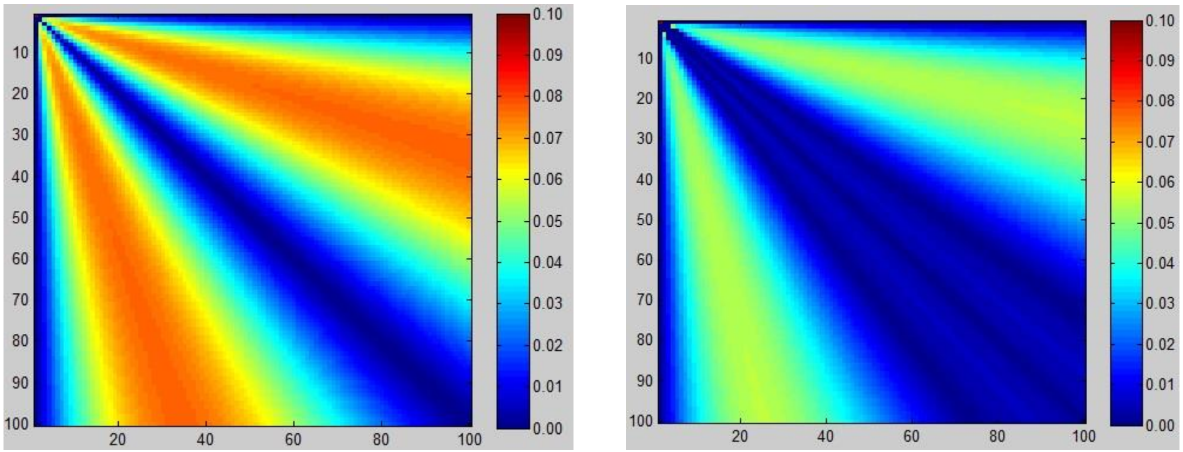
Equation (3) gives a closed form for the finite difference Eikonal solver with a rectangular grid. When $\emptyset$ is equal to $\pi/2$, equation (3) takes the form of the normal Eikonal equation. As examples, the true travel time $t_{2,3}$ is 2.236, with equation (3) is 2.2101 while with Vidale's is 2.2870; for $t_{3,4}$, the true time is 3.6056, with equation (3) is 3.5981 while Vidale's is 3.6103 even if now the direction of wave propagation is closer to an optimal (for Vidale) 45 degrees.

Adaptive finite difference stencils for fast marching

As we saw in the analysis above, we can calculate traveltimes with a stencil that switches between a square grid, i.e. equation (2a) and a rectangular grid, i.e. equation (3), based on the direction of wave propagation. For the rectangular grid, we only use ratios of 1:2 or 2:1 for horizontal and vertical space in order to retain simplicity.

Example

We use a constant velocity with a point source on the top left corner as the model to illustrate our method. The reason for using this simple model is that because we solve finite differences in a conventional Cartesian coordinate system regarding wave propagation through model grids, this model can include different directions of wave propagation related to grid configuration. Therefore, it can provide enough information to illustrate the accuracy of the method. We use Vidale's scheme for comparison because this centered finite difference scheme gives the most accurate result. The parameters of the model are set to the same as that described above. The results of error distribution are shown in Figure 2 and as expected we see that our algorithm improves accuracy.



(a) (b)

Figure 2. Error distribution (a) Vidale's result (b) adaptive stencils' result

Conclusions

During wave propagation, the wavefront is in general neither flat nor circular/spherical and therefore, there is no superior fixed coordinate system for the finite difference eikonal solver. Because finite difference itself is a linear approximation algorithm, higher order finite differences may not guarantee improved accuracy when the wavefront is strongly curved. Since the error of the finite difference eikonal solver depends on the wave propagation direction, the only thing we can do is to design a finite difference scheme that can be robust for all directions of wave propagation. The adaptive finite difference stencils proposed here fulfills this purpose, as verified by the numerical examples presented above.

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