

## Nodal Galerkin Methods for Numerical Modelling of Linear Elasticity with Absorbing Boundary Conditions

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### Summary

We consider here a method for numerical propagation of elastic waves in heterogeneous media based on the weak formulation of the elastodynamic equilibrium equations. The method provides high accuracy in the spatial domain and converges exponentially. It is appropriate for any formulation of the elastic wave equation in any number of spatial dimensions, but for simplicity is only presented here for isotropic media. Absorbing boundary conditions are incorporated into the method analytically and various time-stepping algorithms are investigated. In the conclusion we compare various implementations of the method to several finite-difference algorithms.

### Theory and/or Method

We begin with the *strong* formulation of the elastic wave equation for an arbitrary isotropic heterogeneous medium

$$\text{Find } \mathbf{u} \in C^2(\Omega), \text{ such that } \begin{cases} \rho \ddot{u}_i = \partial_j \sigma_{ij}(\mathbf{u}) + f_i \\ \mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0(\mathbf{x}) \\ \dot{\mathbf{u}}(\mathbf{x}, t = 0) = \mathbf{u}_1(\mathbf{x}) \end{cases}, \forall \mathbf{x} \in \Omega, t \geq 0$$

where stress and strain are related by

$$\sigma_{ij}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u})$$

and summation over repeated indices is assumed. All elastic parameters are allowed to be non-constant spatially-dependent functions, and the forces are taken to be a Ricker wavelet in time and conservative in space. Various models for anisotropy or viscoelasticity can be included in the formulation by changing the stresses. Generally this changes the derivation very little and so we consider only stresses for isotropic inhomogeneous media.

To obtain the *weak* form of the dynamic equilibrium equations we multiply both sides by an arbitrary test function  $v(\mathbf{x})$  and integrate over the entire space  $\Omega$  by parts. This produces

$$\frac{d^2}{dt^2} \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} d\Omega + \int_{\Omega} \sigma_{ij}(\mathbf{u}) \partial_j v_i d\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega + \oint_{\Gamma} \sigma_{ij}(\mathbf{u}) n_j v_i d\Gamma.$$

Absorbing boundary conditions are incorporated into the system by introducing one-way wave equations into the boundary integral. This is done analytically and so no special attention needs to be given to the model definitions at run-time.

The nodal Galerkin method chooses the trial and test functions from those that act like discrete delta functions on a given set of nodes and then replaces differentiation by matrix multiplication and integration by a dot product. This produces a diagonal system of ordinary differential equations representing displacement at a number of spatial nodes at a to-be-determined time.

$$\begin{cases} M\dot{\mathbf{U}}(t) + A\dot{\mathbf{U}}(t) + K\mathbf{U}(t) = \mathbf{F}(t) \\ \mathbf{U}(0) = \mathbf{U}_0 \\ \dot{\mathbf{U}}(0) = \mathbf{U}_1 \end{cases}$$

This is then solved numerically by an appropriate time-stepping algorithm capable of handling both first and second order differentiation simultaneously. In testing we have implemented various finite-difference and Lax-Wendroff methods as well as both implicit and explicit higher-order Runge-Kutta and Adams methods based on reformulating the second-order system as a first-order system in twice as many variables. This has shown that, for time-steps of the order required to image the source wavelet, second-order explicit schemes are usually sufficient, yet, for better resolution, it remains possible to combine an explicit Runge-Kutta method with a higher-order multi-step method.

### Examples

To illustrate the method we propagate a P-wave from a conservative source in 2-dimensions through a sharp jump in the model parameters.

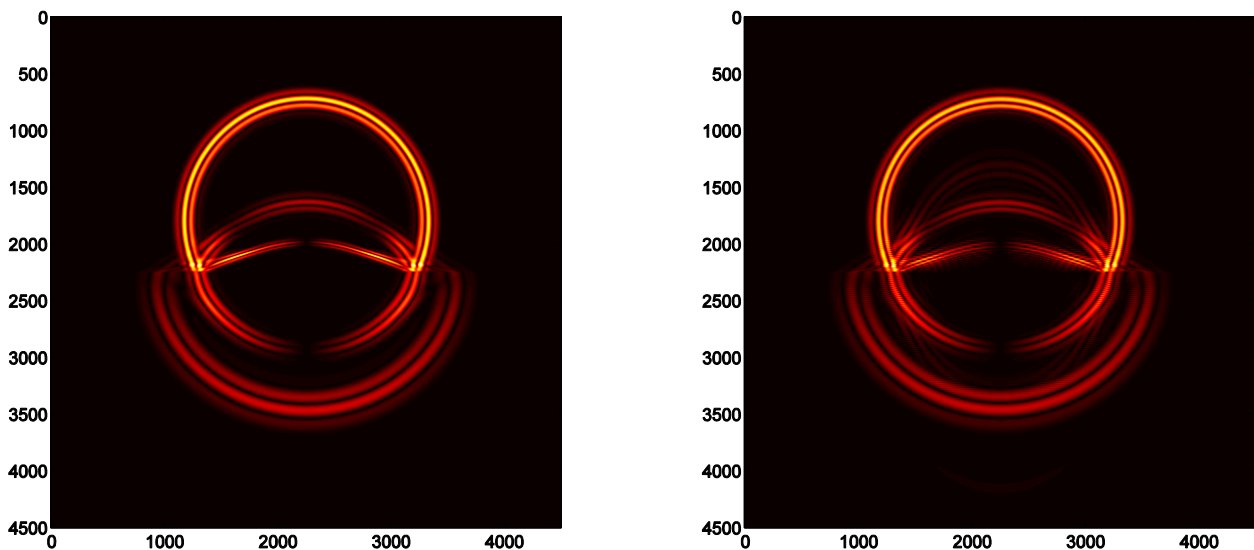


Figure 1: An incident P-wave being converted into reflected and transmitted P and S waves at a discontinuous jump in the material parameters as modeled by a nodal Sinc-Galerkin method (left) and 4<sup>th</sup> order finite-differences (right). Both figures have been clipped to show the numerical dispersion being propagated from the discontinuity.

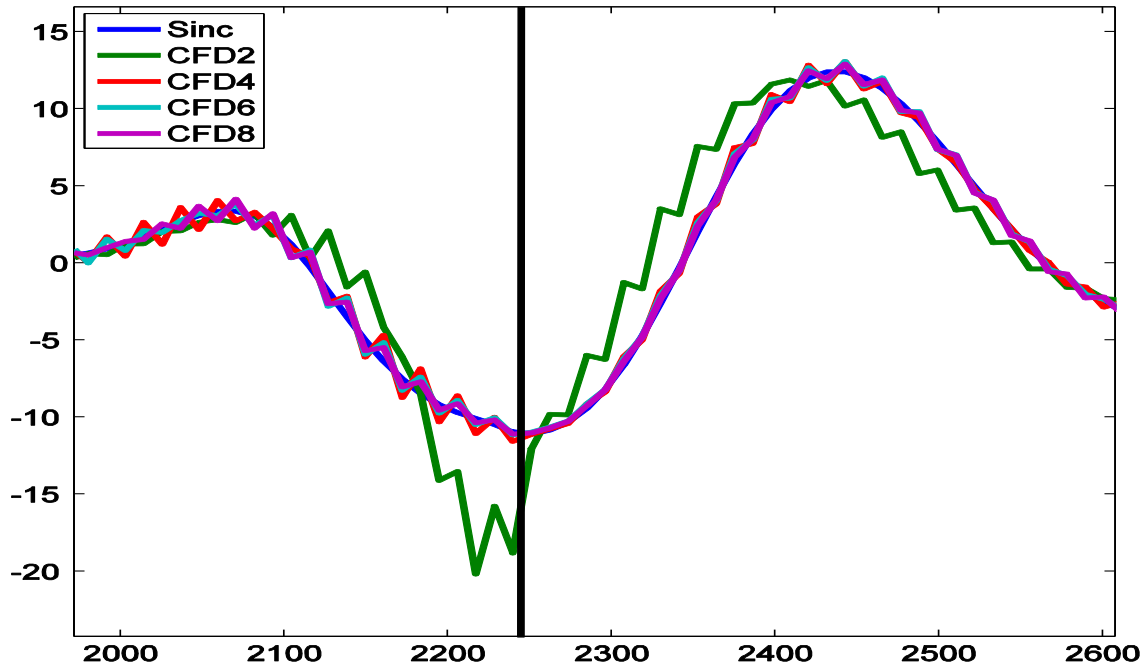


Figure 3: Plot of the vertical component of displacement along the line of symmetry below the source placement for the Sinc-Galerkin method as well as 2<sup>nd</sup>, 4<sup>th</sup>, 6<sup>th</sup>, and 8<sup>th</sup> order finite-differences. Even at 8<sup>th</sup> order the dispersion is present and, even more so, the amount of dispersion does not decrease as the order is increased from 6<sup>th</sup> to 8<sup>th</sup> order. The method does converge to the Sinc-Galerkin solution as the order is increased however.

In testing, first and second-order absorbing boundary conditions have been considered with little noticeable difference found between the two. This agrees with the theory that both conditions are exact for waves incident parallel to the boundary. It's not until the wave-front is almost perpendicular with the boundary that the reflected wave becomes apparent.

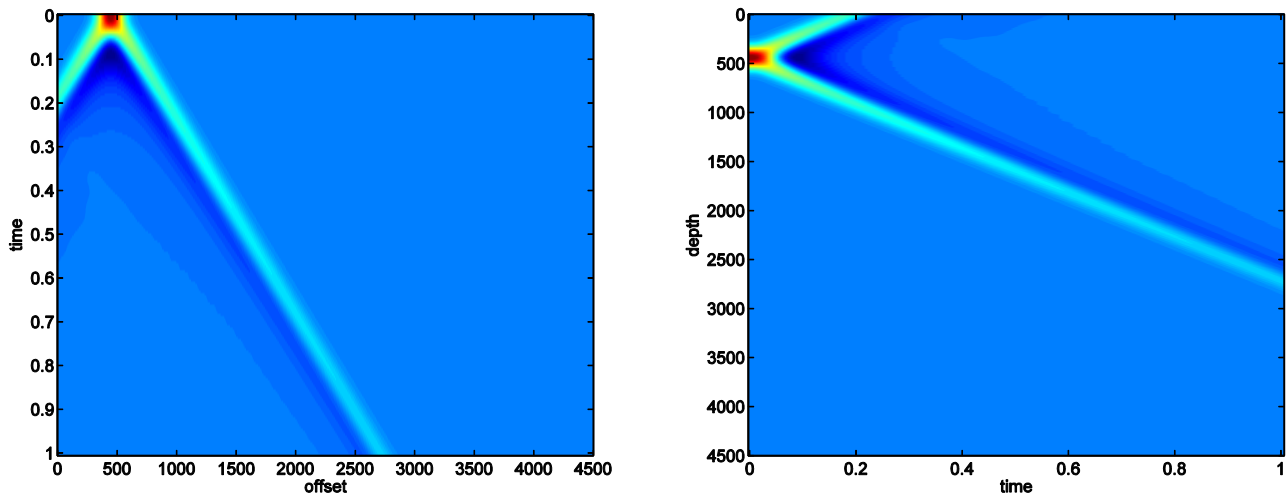


Figure 1: An incident P-wave colliding with the boundary of the numerically truncated region. The left image is a wave hitting a vertical boundary at ninety-degrees and the image on the right is the same situation but for a horizontal boundary.

## Conclusions

The comparison to various finite-difference algorithms shows both convergence and the advantages of the higher-order accuracy in space. The drawback, however, is an increased computational cost to compute the spatial discretization. We found the increased cost to be approximately double that of the 8<sup>th</sup> order finite-difference scheme in terms of wall-clock times. The benefit is that we can accurately image the same wave-fields with a much coarser spatial mesh. This makes the possibility of storing an accurately modelled three-dimensional wave-field, with realistic wavelengths, in RAM on a desktop personal computer much more feasible.

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